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(Continuation of Abstract of Report Number 2 - August 1984 to July 1986)

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Report Number 2

THE THEORY OF DETECTION  
IN INCOMPLETELY CHARACTERIZED NON-GAUSSIAN NOISE

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### Abstract

The problem of detecting a signal known except for amplitude in non-Gaussian noise is addressed. The noise samples are assumed to be independent and identically distributed with a probability density function known except for a few parameters. Using a generalized likelihood ratio test it is proven that for a symmetric noise probability density function the detection performance is asymptotically equivalent to that obtained for a detector designed with a priori knowledge of the noise parameters. A computationally more efficient but equivalent test is proposed and a computer simulation performed to illustrate the theory.

## I. Introduction

A large body of knowledge is available in statistical detection theory for the detection of a known deterministic signal in noise with a known probability density function (PDF).<sup>[5]</sup> In practice, the noise characteristics are never known a priori since they depend on unknown or incompletely understood physical phenomenon.<sup>[6]</sup> Relatively little is known about the design of optimal detectors for this situation. One approach which has been investigated is to use Bayesian methods and assign priors to the unknown parameters of a noise PDF.<sup>[7]</sup> The resulting "optimal" detector requires multidimensional integration which is generally not practical. Furthermore, the performance will be critically dependent on the choice of the priors; the appropriate choice is never known a priori. The entire problem becomes much more difficult when the noise can no longer be characterized as Gaussian.<sup>[8],[9],[10]</sup>

The approach which is taken here is to apply the theory of generalized likelihood ratio testing for composite hypothesis testing.<sup>[1]</sup> The noise PDF is assumed to be known except for a finite set of parameters. The unknown parameters are then estimated using maximum likelihood estimators. This approach does not require the arbitrary selection of priors as in the Bayesian detector. The difficult multidimensional integration required by the Bayesian detector is replaced by a multidimensional maximization of the

likelihood function. Of course, this maximization may be also difficult to implement. By retaining the parametric form of the noise PDF it is expected that the detection performance will exceed that for the optimal nonparametric or robust detector.[11]

Accepting the desirability of a generalized likelihood test (GLRT), several important properties are shown to hold for large data records. The main theorem states that the asymptotic (large data record) performance of the GLRT is equal to that of a clairvoyant GLRT detector if the noise PDF is symmetric. (Examples of symmetric PDFs are the Gaussian, Laplacian and Gaussian mixtures.) The clairvoyant GLRT is one which designs the detector using perfect knowledge of the unknown noise parameters. Hence, the GLRT asymptotically achieves an upper bound in performance and so can be said to be optimal. The theoretical results are quite general and apply to many practical cases of interest. Computer simulation results confirm the asymptotic theory for finite data records. Similar but less general results had been obtained by the author in [12].

The paper is organized as follows. Section II provides a review of the GLRT and its asymptotic properties. The GLRT as applied to the detection problem is described in Section III while its performance and the main theorems are given in Section IV. Section V applies the theory to detection in non-Gaussian noise while Section VI describes the results of a computer simulation.

## II Review of Generalized Likelihood Ratio Testing

This section summarizes the important theory to be used later. The material has been extracted from references [1],[2],[3]. Consider the problem of testing the value of a  $k = r+s$  dimensional parameter

$$\underline{\theta} = \begin{bmatrix} \underline{\theta}_r \\ \underline{\theta}_s \end{bmatrix}$$

where  $\underline{\theta}_r$  is  $r \times 1$  and  $\underline{\theta}_s$  is  $s \times 1$ . A common hypotheses test is that  $\underline{\theta}$  lies in an  $r$  dimensional subspace or [1]

$$H_0: \underline{\theta}_r = \underline{0}, \underline{\theta}_s \quad (1)$$

$$H_1: \underline{\theta}_r \neq \underline{0}, \underline{\theta}_s$$

$\underline{\theta}_s$  is of no concern and may take on any value. It is sometimes referred to as a nuisance parameter. Assuming the data  $\underline{x} = [x_0 \ x_1 \ \dots \ x_{N-1}]^T$  are observed with a joint probability density function  $p(\underline{x}; \underline{\theta}_r, \underline{\theta}_s)$ , a generalized likelihood ratio test for testing (1) is to decide  $H_1$  if

$$L_G = \frac{p(\underline{x}; \hat{\underline{\theta}}_r, \hat{\underline{\theta}}_s)}{p(\underline{x}; \underline{0}, \hat{\underline{\theta}}_s)} > \gamma \quad (2)$$

for some threshold  $\gamma$ .  $\hat{\underline{\theta}}_s$  is the maximum likelihood estimator (MLE) of  $\underline{\theta}_s$  assuming  $H_0$  is true while  $\hat{\underline{\theta}}_r, \hat{\underline{\theta}}_s$  are the joint MLEs of  $\underline{\theta}_r$  and  $\underline{\theta}_s$  assuming  $H_1$  is true.  $\hat{\underline{\theta}}_s$  is found by maximizing  $p(\underline{x}; \underline{0}, \underline{\theta}_s)$  over  $\underline{\theta}_s$ .

Likewise,  $\hat{\underline{\theta}}_r, \hat{\underline{\theta}}_s$  are found by maximizing  $p(\underline{x} ; \underline{\theta}_r, \underline{\theta}_s)$  over  $\underline{\theta}_r, \underline{\theta}_s$ .

The statistics of  $L_G$  are difficult to obtain in general. For large data records ( $N$  large) or asymptotically it may be shown that  $2 \ln L_G$  is distributed in the following manner. [1], [2]

$$2 \ln L_G \sim \chi_r^2 \quad \text{under } H_0 \quad (3a)$$

$$2 \ln L_G \sim \chi'^2(\lambda, r) \quad \text{under } H_1 \quad (3b)$$

Here  $\chi_r^2$  represents a Chi squared distribution with  $r$  degrees of freedom and  $\chi'^2(\lambda, r)$  represents a noncentral Chi square distribution with  $r$  degrees of freedom and noncentrality parameter  $\lambda$ . Note that  $\chi'^2(0, r) = \chi_r^2$  or the distribution under  $H_0$  is a special case of the distribution under  $H_1$  and occurs when  $\lambda = 0$ . The noncentrality parameter  $\lambda$ , which is a measure of the discrimination between the two hypotheses, is

$$\lambda = \underline{\theta}_r^T \left[ \underline{I}_{\underline{\theta}_r \underline{\theta}_r}(\underline{0}, \underline{\theta}_s) - \underline{I}_{\underline{\theta}_r \underline{\theta}_s}(\underline{0}, \underline{\theta}_s) \underline{I}_{\underline{\theta}_s \underline{\theta}_s}^{-1}(\underline{0}, \underline{\theta}_s) \underline{I}_{\underline{\theta}_r \underline{\theta}_s}^T(\underline{0}, \underline{\theta}_s) \right] \underline{\theta}_r \quad (4)$$

where  $\underline{\theta}_r, \underline{\theta}_s$  are the true values. The terms in the brackets of (4) are found by partitioning the Fisher information matrix for  $\underline{\theta}$

$$\underline{I}(\underline{\theta}) = \begin{bmatrix} \underline{I}_{\underline{\theta}_r \underline{\theta}_r}(\underline{\theta}_r, \underline{\theta}_s) & \underline{I}_{\underline{\theta}_r \underline{\theta}_s}(\underline{\theta}_r, \underline{\theta}_s) \\ \underline{I}_{\underline{\theta}_s \underline{\theta}_r}(\underline{\theta}_r, \underline{\theta}_s) & \underline{I}_{\underline{\theta}_s \underline{\theta}_s}(\underline{\theta}_r, \underline{\theta}_s) \end{bmatrix} \quad (5)$$

and the partitions are defined as

$$\begin{aligned}
\mathbf{I}_{\theta_r \theta_r}(\underline{\theta}_r, \underline{\theta}_s) &= E \left[ \left( \frac{\partial \ln p}{\partial \underline{\theta}_r} \right) \left( \frac{\partial \ln p}{\partial \underline{\theta}_r} \right)^T \right] & r \times r \\
\mathbf{I}_{\theta_r \theta_s}(\underline{\theta}_r, \underline{\theta}_s) &= E \left[ \left( \frac{\partial \ln p}{\partial \underline{\theta}_r} \right) \left( \frac{\partial \ln p}{\partial \underline{\theta}_s} \right)^T \right] & r \times s \\
\mathbf{I}_{\theta_s \theta_r}(\underline{\theta}_r, \underline{\theta}_s) &= \mathbf{I}_{\theta_r \theta_s}^T(\underline{\theta}_r, \underline{\theta}_s) & s \times r \\
\mathbf{I}_{\theta_s \theta_s}(\underline{\theta}_r, \underline{\theta}_s) &= E \left[ \left( \frac{\partial \ln p}{\partial \underline{\theta}_s} \right) \left( \frac{\partial \ln p}{\partial \underline{\theta}_s} \right)^T \right] & s \times s
\end{aligned} \tag{6}$$

All the partitions of the Fisher information matrix are evaluated at  $\underline{\theta}_r = \underline{0}$  and the true value of  $\underline{\theta}_s$  for use in (4).

The motivation for using a GLRT is that for large data records it exhibits certain optimality properties. (It should be noted that in general the hypothesis testing problem of (1) does not admit a uniformly most powerful (UMP) test.<sup>[1]</sup>) These optimality properties are that of all the tests which are invariant to a natural set of transformations the GLRT exhibits the largest probability of detection for all values of the unknown parameters. The GLRT is said to be the asymptotically uniformly most powerful invariant (UMPI) test.<sup>[4]</sup> The conditions under which the asymptotic results apply to finite length data records are difficult to quantify in general. Heuristically, it may be said that the asymptotic results will apply if

- 1) The MLEs necessary to define the GLRT are adequately characterized by their asymptotic properties or the PDF of the MLE is Gaussian with mean equal to the true parameter value and covariance matrix equal to the inverse of the Fisher information matrix and

- 2) The value of  $\underline{\theta}_r$  when  $H_1$  is true is close to the value when  $H_0$  is true or  $\underline{\theta}_r \approx \underline{0}$ . In effect, it says that we must be testing for slight departures of  $\underline{\theta}_r$  from zero.

To implement the GLRT requires one to find the MLEs under  $H_0$  and  $H_1$ . In many cases this is a difficult analytical task. To avoid some of the difficulties use can be made of the Rao test which is asymptotically equivalent to the GLRT. The Rao test [2], [3] decides  $H_1$  if

$$L_R = \frac{\frac{\partial \ln p(\underline{x}; \underline{\theta}_r, \hat{\underline{\theta}}_s)}{\partial \underline{\theta}_r} \bigg|_{\underline{\theta}_r = \underline{0}}^T}{\underline{V}(\underline{0}, \hat{\underline{\theta}}_s) \frac{\partial \ln p(\underline{x}; \underline{\theta}_r, \hat{\underline{\theta}}_s)}{\partial \underline{\theta}_r} \bigg|_{\underline{\theta}_r = \underline{0}}} > \eta \quad (7)$$

where

$$\underline{V}(\underline{\theta}_r, \underline{\theta}_s) = \left[ \underline{I}_{\underline{\theta}_r \underline{\theta}_r}(\underline{\theta}_r, \underline{\theta}_r) - \underline{I}_{\underline{\theta}_r \underline{\theta}_s}(\underline{\theta}_r, \underline{\theta}_s) \underline{I}_{\underline{\theta}_s \underline{\theta}_s}^{-1}(\underline{\theta}_r, \underline{\theta}_s) \underline{I}_{\underline{\theta}_r \underline{\theta}_s}^T(\underline{\theta}_r, \underline{\theta}_s) \right]$$

Note that only the MLE of  $\underline{\theta}_s$  under  $H_0$  need be found. The MLE of  $\underline{\theta}_r$  and  $\underline{\theta}_s$  under  $H_1$  are no longer required. Also, the statistics of  $L_R$  are given by (3).

### III Statement of Problem and Detector

We will consider the problem of detecting a signal known except for amplitude in independent and identically distributed non-Gaussian noise. The univariate PDF of the noise is assumed to be known except for a finite set of parameters. Mathematically, we have

$$H_0: x_t = n_t \quad (8)$$

$$H_1: x_t = \mu s_t + n_t$$

for  $t = 0, 1, \dots, N-1$ .  $\mu$ , the amplitude, is unknown and can take on any value while  $s_t$  is known. The PDF of  $n_t$ , which is the same for any  $t$  is denoted by  $p(n; \underline{a})$ , where  $\underline{a}$  is a set of unknown parameters. The PDF for the noise is not restricted to be Gaussian, only that it satisfies certain regularity conditions to insure the validity of the asymptotic MLE properties. [3]

The problem of (8) can be recast as a test of the amplitude

$$H_0: \mu = 0, \underline{a} \quad (9)$$

$$H_1: \mu \neq 0, \underline{a}$$

which is recognized as a composite hypothesis testing problem of the form of (1). Because  $\mu$  may take on positive or negative values it is well known that no UMP test exists, even for a priori knowledge of  $\underline{a}$ . [1] Hence a GLRT is proposed. Due to the independence of the noise the GLRT of (2) reduces to

$$L_G = \frac{\prod_{t=0}^{N-1} p(x_t - \hat{\mu} s_t; \hat{\underline{a}})}{\prod_{t=0}^{N-1} p(x_t; \hat{\underline{a}})} \quad (10)$$

where the identification  $\underline{\theta} = [\underline{\theta}_r^T \underline{\theta}_s^T]^T = [\mu \ \underline{a}^T]^T$  has been made.  $\hat{\mu}$  is the MLE of  $\mu$  under  $H_1$  and  $\hat{\underline{a}}, \hat{\underline{a}}$  are the MLEs of  $\underline{a}$  under  $H_1$  and  $H_0$ , respectively.  $\hat{\mu}, \hat{\underline{a}}$  are found by maximizing

$$\prod_{t=0}^{N-1} p(\mathbf{x}_t - \mu \mathbf{s}_t ; \underline{a})$$

over  $\mu$  and  $\underline{a}$  while  $\hat{\underline{a}}$  is found by maximizing

$$\prod_{t=0}^{N-1} p(\mathbf{x}_t ; \underline{a})$$

over  $\underline{a}$ .

#### IV Asymptotic Performance of GLRT

The asymptotic performance of the GLRT given in (10) may be found by making use of the results of Section II. Making the identifications

$$\underline{\theta} = \begin{bmatrix} \underline{\theta}_r \\ \underline{\theta}_s \end{bmatrix} = \begin{bmatrix} \mu \\ \underline{a} \end{bmatrix}$$

so that  $\underline{\theta}_r$  is a scalar or  $r = 1$  and  $\underline{a}$  is an  $s \times 1$  vector, we have from (3), (4)

$$2 \ln L_G \sim \chi_1^2 \quad \text{under } H_0 \quad (11)$$

$$\sim \chi'^2(1, \lambda) \quad \text{under } H_1$$

$$\text{where } \lambda = \mu^2 \left[ \underline{I}_{\mu\mu}(0, \underline{a}) - \underline{I}_{\mu\alpha}(0, \underline{a}) \underline{I}_{\alpha\alpha}^{-1}(\underline{a}, \underline{a}) \underline{I}_{\mu\alpha}^T(0, \underline{a}) \right] \quad (12)$$

The probability of detection

$$P_D = \Pr \{ 2 \ln L_G > \gamma' \mid H_1 \} \quad (13)$$

and the probability of false alarm

$$P_{FA} = \Pr \{ 2 \ln L_G > \gamma' \mid H_0 \} \quad (14)$$

are easily found by noting that a  $\chi'^2(1, \lambda)$  random variable is equivalent to the square of a normal random variable with mean  $\sqrt{\lambda}$  and variance 1. [1] It can be shown that

$$P_{FA} = 2Q(\sqrt{\gamma'}) \quad (15)$$

$$P_D = Q(\sqrt{\gamma'} - \sqrt{\lambda}) + Q(\sqrt{\gamma'} + \sqrt{\lambda}) \quad (16)$$

$$\text{where } Q(\xi) = \int_{\xi}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

It is interesting to note that the probability of detection increases monotonically with  $\lambda$ . Furthermore a special case of the theory presented in Section II occurs when  $\underline{\theta}_s$ , the nuisance parameters, are known. For the problem at hand  $\underline{\theta}_s = \underline{a}$ . If  $\underline{a}$  is known, then the clairvoyant GLRT as given by (10) becomes

$$L_{GC} = \frac{\prod_{t=0}^{N-1} p(x_t - \hat{\mu} s_t; \underline{a})}{\prod_{t=0}^{N-1} p(x_t; \underline{a})} \quad (17)$$

and the asymptotic statistics of  $2 \ln L_{GC}$  are given by (11) but with

$$\lambda_c = \mu^2 I_{\mu\mu}(0, \underline{a}) \quad (18)$$

Note that the performance of the GRLT when  $\underline{a}$  is known is always better than or at worst equal to the performance when  $\underline{a}$  is unknown since  $\lambda_c \geq \lambda$ . The loss in performance when  $\underline{a}$  is unknown may be attributed to the reduction in Fisher information due to having to jointly estimate  $\mu$  and  $\underline{a}$ , even though  $\underline{a}$  is of no concern. If it happens that  $\lambda_c = \lambda$  then the GLRT will produce the same detection performance as the clairvoyant GLRT detector, i.e., the GLRT which incorporates a priori knowledge of the unknown parameter  $\underline{a}$ . From examination of (12) and (18) this will be the case if

$$I_{\mu\alpha}(0, \underline{a}) = 0 \quad (19)$$

This condition is equivalent to requiring the MLEs of  $\mu$  and  $\underline{a}$  to be asymptotically uncorrelated (and hence independent) when  $\mu$  is close to zero.

An example which illustrates this behavior is the following classical problem. Consider the detection problem of (8) with  $s_t = 1$

and  $n_t \sim N(0, \sigma^2)$ . The noise samples are independent and identically distributed and the variance  $\sigma^2$  is unknown. The amplitude  $\mu$  can be either positive or negative so that no UMP test exists. The nuisance parameter in this problem is the noise variance  $\sigma^2$ . If one derives the GLRT, (see Appendix A) then the test statistic is

$$2\ln L_G = N \ln \left[ 1 - \frac{\bar{x}^2}{\frac{1}{N} \sum_{t=0}^{N-1} x_t^2} \right] \quad (20)$$

where  $\bar{x} = \frac{1}{N} \sum_{t=0}^{N-1} x_t$ , which is the sample mean.

If however  $\sigma^2$  were known, then the GLRT would yield

$$2\ln L_{GC} = \left[ \frac{\bar{x}}{\sigma/\sqrt{N}} \right]^2 \quad (21)$$

which might be referred to as a clairvoyant detector since it makes use of a priori knowledge of  $\sigma^2$ . Now assume  $N \rightarrow \infty$  so that by the law of large numbers  $\bar{x} \rightarrow \mu$  and assume  $\mu$  is close to zero. Then from (20)  $2\ln L_G$  converges to

$$2\ln L_G \rightarrow \frac{N\bar{x}^2}{\frac{1}{N} \sum_{t=0}^{N-1} x_t^2} \quad (22)$$

using the relation  $\ln(1-x) \approx -x$  for  $x \ll 1$ . But by Slutsky's Theorem  $2\ln L_G$  converges in distribution to that of  $2\ln L_{GC}$ . Recall that it is

assumed  $\mu \rightarrow 0$  and hence

$$\frac{1}{N} \sum_{t=0}^{N-1} x_t^2 \rightarrow \sigma^2$$

whether or not a signal is present. Since the distributions of  $L_G$  and  $L_{GC}$  are asymptotically identical, their performances are identical. It is an easy matter to show that (19) holds or

$$I_{\mu\sigma^2}(0, \sigma^2) = 0 \quad (23)$$

(Actually  $I_{\mu\sigma^2}(\mu, \sigma^2) = 0$  for any  $\mu$  in this case.) A careful examination of the form of  $I_{\mu\sigma^2}(\mu, \sigma^2)$  reveals that the symmetric nature of the Gaussian PDF is responsible for (23). In fact as is now proven  $I_{\mu\alpha}(\mu, \underline{\alpha})$  will be zero whenever the PDF of the noise is an even function.

Theorem 1: Assume that  $N$  samples of a process  $x_t = \mu s_t + n_t$  are observed, where the noise samples are independent and each one is distributed according to the PDF  $p(n; \underline{\alpha})$  which depends on the parameter  $\underline{\alpha}$ .  $s_t$ , the signal, is known. If  $p(-n; \underline{\alpha}) = p(n; \underline{\alpha})$ , for all  $\underline{\alpha}$ , then  $I_{\mu\alpha}(\mu, \underline{\alpha}) = 0$ , where the Fisher information is based on  $\{x_0, x_1, \dots, x_{N-1}\}$ .

Proof: By the definition of the Fisher information

$$\underline{I}_{\mu\alpha}(\mu, \underline{a}) = E \left[ \frac{\partial \ln \prod_{t=0}^{N-1} p(x_t - \mu s_t; \underline{a})}{\partial \mu} \quad \frac{\partial \ln \prod_{t=0}^{N-1} p(x_t - \mu s_t; \underline{a})}{\partial \underline{a}} \right] \quad (24)$$

Since the noise samples are independent,  $p(x_i - \mu s_i; \underline{a})$  is independent of  $p(x_j - \mu s_j; \underline{a})$  for  $j \neq i$  and each term

$$\frac{\partial \ln p(x_t - \mu s_t; \underline{a})}{\partial \theta}$$

where  $\theta = \mu$  or  $\alpha_i$  has expectation zero. Hence,

$$\underline{I}_{\mu\alpha}(\mu, \underline{a}) = \sum_{t=0}^{N-1} E \left[ \frac{\partial \ln p(x_t - \mu s_t; \underline{a})}{\partial \mu} \quad \frac{\partial \ln p(x_t - \mu s_t; \underline{a})}{\partial \underline{a}} \right] \quad (25)$$

Consider a single term in the summation, which may be written as

$$\int_{-\infty}^{\infty} \frac{\partial \ln p(x_t - \mu s_t; \underline{a})}{\partial \mu} \quad \frac{\partial \ln p(x_t - \mu s_t; \underline{a})}{\partial \underline{a}} \quad p(x_t - \mu s_t; \underline{a}) dx_t$$

Using the chain rule this becomes

$$-s_t \int_{-\infty}^{\infty} \frac{\partial \ln p(x_t - \mu s_t; \underline{a})}{\partial (x_t - \mu s_t)} \quad \frac{\partial \ln p(x_t - \mu s_t; \underline{a})}{\partial \underline{a}} \quad p(x_t - \mu s_t; \underline{a}) dx_t$$

Letting  $n_t = x_t - \mu s_t$ , we have

$$-s_t \int_{-\infty}^{\infty} \frac{\partial \ln p(n_t; \underline{a})}{\partial n_t} \frac{\partial \ln p(n_t; \underline{a})}{\partial \underline{a}} p(n_t; \underline{a}) dn_t$$

That this last expression equals zero follows from the assumption that  $p(n_t; \underline{a})$  is an even function in  $n_t$ . For  $p(n_t; \underline{a})$  even

$$\frac{\partial \ln p(n_t; \underline{a})}{\partial \underline{a}}$$

is even in  $n_t$  and

$$\frac{\partial \ln p(n_t; \underline{a})}{\partial n_t}$$

is an odd function in  $n_t$ . The integral is then an odd function which is integrated over an even interval resulting in zero. To prove the even property of  $\partial \ln p(n_t; \underline{a}) / \partial \underline{a}$  consider the  $i$ th component

$$\frac{\partial \ln p(n_t; \underline{a})}{\partial a_i} = \lim_{\Delta_i \rightarrow 0} \frac{\ln p(n_t; \underline{a} + \underline{\Delta}_i) - \ln p(n_t; \underline{a})}{\Delta_i}$$

where  $\underline{\Delta}_i$  is a vector with all zeros except for the  $i$ th element which is the small increment  $\Delta_i$ . It is clear from examination of the partial derivative definition that if the PDF is an even function for all  $\underline{a}$ , then  $\partial \ln p(n_t; \underline{a}) / \partial a_i$  is also even. Next consider

$$\begin{aligned}
\frac{\partial \ln p(n_t; \underline{a})}{\partial n_t} &= \lim_{\Delta \rightarrow 0} \frac{\ln p(n_t + \Delta; \underline{a}) - \ln p(n_t; \underline{a})}{\Delta} \\
\frac{\partial \ln p(-n_t; \underline{a})}{\partial n_t} &= \lim_{\Delta \rightarrow 0} \frac{\ln p(-n_t + \Delta; \underline{a}) - \ln p(-n_t; \underline{a})}{\Delta} \\
&= \lim_{\Delta \rightarrow 0} \frac{\ln p(n_t - \Delta; \underline{a}) - \ln p(n_t; \underline{a})}{\Delta} \\
&= - \lim_{\Delta \rightarrow 0} \frac{\ln p(n_t; \underline{a}) - \ln p(n_t - \Delta; \underline{a})}{\Delta} \\
&= - \frac{\partial \ln p(n_t; \underline{a})}{\partial n_t}
\end{aligned}$$

The last step follows from the definition of the derivative which states that the same limit must be obtained for an approach from the right or the left. Hence  $\partial \ln p(n_t; \underline{a}) / \partial n_t$  is an odd function as asserted. Since each term in the sum of (25) has been shown to be zero, the conclusion of the theorem that  $I_{\mu \underline{a}}(\mu, \underline{a}) = 0$  follows.

Theorem 2: Assume that  $N$  samples of a process  $x_t = \mu s_t + n_t$  are observed, where the noise samples are independent and each one is distributed according to a symmetric PDF,  $p(n; \underline{a})$ . The PDF of the noise depends upon a vector  $\underline{a}$  of unknown parameters. Then the GLRT (10) for testing  $\mu = 0$  vs  $\mu \neq 0$  has asymptotic performance given by (15) and (16) with  $\lambda$  given by (18) and the performance is as good as the performance of the clairvoyant GLRT ( $\underline{a}$  known) as given by (17).

Proof: follows directly from previous discussion.

The next theorem concerns the implementation of the GLRT via the Rao test.

Theorem 3: All statements hold true in Theorem 2 if the GLRT is replaced by a Rao test which is defined as

$$L_R = \frac{\left[ \sum_{t=0}^{N-1} \frac{\partial \ln p(\mathbf{x}_t - \mu \mathbf{s}_t; \hat{\underline{a}})}{\partial \mu} \bigg|_{\mu = 0} \right]^2}{I_{\mu\mu}(0, \hat{\underline{a}})} > \eta \quad (26)$$

where  $\hat{\underline{a}}$  is the MLE under  $H_0$  and

$$I_{\mu\mu}(0, \underline{a}) = \sum_{t=0}^{N-1} s_t^2 \left[ \int_{-\infty}^{\infty} \left( \frac{p'(n; \underline{a})}{p(n; \underline{a})} \right)^2 p(n; \underline{a}) dn \right] \quad (27)$$

$p'(n; \underline{a})$  denotes the derivative with respect to  $n$ . The quantity in brackets is the Fisher information for a shift in mean or the intrinsic accuracy of a PDF. [2]

Proof: The asymptotic equivalence of the Rao test to the GLRT is proven in [3]. We now prove that the Rao test for our problem is given by (26) and (27). Starting with (7) let  $\underline{\theta}_r = \mu$  and  $\underline{\theta}_s = \underline{a}$ . Also note that  $I_{\mu\mu}(\mu, \underline{a}) = 0$ . Then,

$$L_R = \frac{\left[ \left. \frac{\partial \ln p(\underline{x}; \mu, \underline{\hat{a}})}{\partial \mu} \right|_{\mu = 0} \right]^2}{I_{\mu\mu}(0, \underline{\hat{a}})}$$

$I_{\mu\mu}(\mu, \underline{a})$  is found as follows.

$$\begin{aligned} I_{\mu\mu}(\mu, \underline{a}) &= E \left[ \left( \frac{\sum_{t=0}^{N-1} \frac{\partial \ln p(\underline{x}_t - \mu s_t; \underline{a})}{\partial \mu}}{\partial \mu} \right)^2 \right] \\ &= E \left[ \left( \sum_{t=0}^{N-1} \frac{\partial \ln p(\underline{x}_t - \mu s_t; \underline{a})}{\partial \mu} \right)^2 \right] \\ &= \sum_{t=0}^{N-1} E \left[ \left( \frac{\partial \ln p(\underline{x}_t - \mu s_t; \underline{a})}{\partial \mu} \right)^2 \right] \\ &= \sum_{t=0}^{N-1} s_t^2 E \left[ \left( \frac{\partial \ln p(\underline{n}_t; \underline{a})}{\partial \underline{n}_t} \right)^2 \right] \\ &= \sum_{t=0}^{N-1} s_t^2 \int_{-\infty}^{\infty} \left[ \frac{p'(\underline{n}_t; \underline{a})}{p(\underline{n}_t; \underline{a})} \right]^2 p(\underline{n}_t; \underline{a}) d\underline{n}_t \\ &= \sum_{t=0}^{N-1} s_t^2 \int_{-\infty}^{\infty} \left[ \frac{p'(\underline{n}; \underline{a})}{p(\underline{n}; \underline{a})} \right]^2 p(\underline{n}; \underline{a}) d\underline{n} \end{aligned}$$

which is independent of  $\mu$  and hence  $I_{\mu\mu}(0, \underline{a})$  follows.

To avoid the integration required to compute  $I_{\mu\mu}(0, \underline{a})$  it is

possible to use the asymptotic equivalence

$$I_{\mu\mu}(\mu, \underline{a}) = \sum_{t=0}^{N-1} s_t^2 E \left[ \left( \frac{\partial \ln p(n_t; \underline{a})}{\partial n_t} \right)^2 \right]$$

$$\approx \frac{1}{N} \sum_{n=0}^{N-1} s_n^2 \sum_{t=0}^{N-1} \left[ \frac{p'(n_t; \underline{a})}{p(n_t; \underline{a})} \right]^2$$

which for  $\mu = 0$  (so that  $x_t = n_t$ ) becomes

$$I(0, \underline{a}) \approx \frac{1}{N} \sum_{n=0}^{N-1} s_n^2 \sum_{t=0}^{N-1} \left[ \frac{p'(x_t; \underline{a})}{p(x_t; \underline{a})} \right]^2 \quad (28)$$

This result follows from the law of large numbers. In essence, the integral of (27) is replaced by a Monte Carlo evaluation. Rao's test becomes

$$L'_R = \frac{\left[ \sum_{t=0}^{N-1} \frac{\partial \ln p(x_t - \mu s_t; \hat{\underline{a}})}{\partial \mu} \right]_{\mu=0}^2}{\frac{1}{N} \sum_{n=0}^{N-1} s_n^2 \sum_{t=0}^{N-1} \left[ \frac{p'(x_t; \hat{\underline{a}})}{p(x_t; \hat{\underline{a}})} \right]^2} \quad (29)$$

The asymptotic statistics of  $L'_R$  are identical to those of  $L_R$  and  $2 \ln L_G$ . In the next section an example of the use of  $L'_R$  is given.

## V Application to Detection in Non-Gaussian Noise

Consider the problem of (8) when  $n_t$  has a Gaussian mixture PDF or

$$p(n) = \frac{1-\varepsilon}{\sqrt{2\pi} \sigma_B} e^{-\frac{n^2}{2\sigma_B^2}} + \frac{\varepsilon}{\sqrt{2\pi} \sigma_I} e^{-\frac{n^2}{2\sigma_I^2}} \quad (30)$$

where  $\sigma_I \gg \sigma_B$ . The first term represents the usual background noise while the second term is an interference component. As an example a realization of the noise time series for  $\sigma_B^2 = 1$ ,  $\sigma_I^2 = 100$  and  $\varepsilon = 0.1$  is shown in Figure 1. The noise spikes are obviously due to the contaminating component. The unknown noise PDF parameter is assumed to be the mixture parameter  $\varepsilon$ . It is assumed that the noise variances  $\sigma_B^2$  and  $\sigma_I^2$  are known. The known signal  $s_t$  is assumed to be a DC signal with amplitude one so that the problem is to detect a DC signal  $\mu$  of unknown level in independent and identically distributed contaminated Gaussian noise with an unknown mixture parameter  $\varepsilon$ . The noise PDF is symmetric for all  $\varepsilon$  in the range  $[0,1]$  so that the results of Theorems 1-3 apply. The Rao test of (29) becomes

$$L'_R = \frac{\left[ \sum_{t=0}^{N-1} \frac{\partial \ln p(x_t - \mu; \hat{\varepsilon})}{\partial \mu} \right]_{\mu=0}^2}{\sum_{t=0}^{N-1} \left[ \frac{p'(x_t; \hat{\varepsilon})}{p(x_t; \hat{\varepsilon})} \right]^2} \quad (31)$$

where  $\hat{\varepsilon}$  is the MLE of  $\varepsilon$  assuming  $H_0$  is true or  $\hat{\varepsilon}$  is the value of  $\varepsilon$  over

the interval  $[0,1]$  which maximizes

$$\prod_{t=0}^{N-1} \left[ (1 - \varepsilon) \phi_B(x_t) + \varepsilon \phi_I(x_t) \right] \quad (32)$$

$$\text{where } \phi_B(n) = \frac{1}{\sqrt{2\pi} \sigma_B} e^{-\frac{1}{2}(n/\sigma_B)^2} \text{ and } \phi_I(n) = \frac{1}{\sqrt{2\pi} \sigma_I} e^{-\frac{1}{2}(n/\sigma_I)^2}$$

Some simplifications to (31) are possible.

$$\begin{aligned} & \left. \frac{\partial \ln p(x_t - \mu; \hat{\varepsilon})}{\partial \mu} \right|_{\mu=0} \\ &= \frac{\left[ -(1 - \hat{\varepsilon}) \left( \frac{x_t - \mu}{\sigma_B^2} \right) \phi_B(x_t - \mu) - \hat{\varepsilon} \left( \frac{x_t - \mu}{\sigma_I^2} \right) \phi_I(x_t - \mu) \right] \Big|_{\mu=0}}{p(x_t - \mu; \hat{\varepsilon})} \\ &= \frac{\left[ (1 - \hat{\varepsilon}) (x_t/\sigma_B^2) \phi_B(x_t) + \hat{\varepsilon} (x_t/\sigma_I^2) \phi_I(x_t) \right]}{(1 - \hat{\varepsilon}) \phi_B(x_t) + \hat{\varepsilon} \phi_I(x_t)} \end{aligned}$$

so that

$$L'_R = \frac{\left[ \sum_{t=0}^{N-1} \frac{p'(x_t; \hat{\varepsilon})}{p(x_t; \hat{\varepsilon})} \right]^2}{\sum_{t=0}^{N-1} \left[ \frac{p'(x_t; \hat{\varepsilon})}{p(x_t; \hat{\varepsilon})} \right]^2} \quad (33)$$

$$\text{where } p(x_t; \hat{\varepsilon}) = (1 - \hat{\varepsilon}) \phi_B(x_t) + \hat{\varepsilon} \phi_I(x_t)$$

$$p'(x_t; \hat{\varepsilon}) = -[(1 - \hat{\varepsilon}) (x_t/\sigma_B^2) \phi_B(x_t) + \hat{\varepsilon} (x_t/\sigma_I^2) \phi_I(x_t)]$$

The asymptotic performance of  $L_R'$  is given by (15) and (16) where

$$\lambda = \mu^2 I_{\mu\mu}(0, \varepsilon)$$

From (27) with  $s_t = 1$

$$\lambda = N\mu^2 \int_{-\infty}^{\infty} \left[ \frac{p'(n; \varepsilon)}{p(n; \varepsilon)} \right]^2 p(n; \varepsilon) dn \quad (34)$$

The noncentrality parameter  $\lambda$  cannot be evaluated in closed form so that a numerical evaluation is required.

## VI Computer Simulation Results

The performance of the detector described in previous section is now examined via a computer simulation. To summarize the assumptions the detection problem addressed is

$$H_0: x_t = n_t$$

$$H_1: x_t = \mu + n_t$$

for  $t = 0, 1, \dots, N-1$ .  $N$  was chosen to be 500. The signal level  $\mu$  is unknown and can be positive or negative. The noise samples  $n_t$  are independent and identically distributed with PDF

$$p(n) = \frac{1-\varepsilon}{\sqrt{2\pi}\sigma_B} e^{-\frac{1}{2}(n/\sigma_B)^2} + \frac{\varepsilon}{\sqrt{2\pi}\sigma_I} e^{-\frac{1}{2}(n/\sigma_I)^2}$$

The noise variances  $\sigma_B^2$  and  $\sigma_I^2$  are assumed to be known and  $\sigma_B^2 = 1$ ,  $\sigma_I^2 = 100$  for purposes of the simulation. The mixture parameter  $\varepsilon$  is assumed to be unknown and  $\varepsilon = 0.1$ . The SNR is defined to be

$$\text{SNR} = 10 \log_{10} \frac{N\mu^2}{\sigma_B^2} \text{ dB} \quad (35)$$

which represents the output SNR of a matched filter.  $\mu$  is set for each SNR by using (35). The Rao test is given by (33). A clairvoyant detector is defined to be identical to (33) except that the known value of  $\varepsilon$  is used in place of its MLE under  $H_0$ . The theoretical asymptotic performance of the Rao test and the clairvoyant detector are given by (15) and (16) with  $\lambda$  given by (34). For a  $P_{FA}$  of  $10^{-2}$  the simulation results for the Rao test and the clairvoyant test as well as the theoretical asymptotic performance are shown in Figure 2. As expected the theoretical asymptotic results appear to accurately predict performance. The Rao test performance is nearly identical to that of the clairvoyant test in accordance with Theorem 3.

As a benchmark to which performance might be compared it is illustrative to consider a matched filter. The matched filter which would result if the contaminating Gaussian PDF were ignored is to compare the sample mean to a threshold or

$$\frac{1}{N} \sum_{t=0}^{N-1} x_t > \alpha$$

The matched filter requires the additional knowledge of the sign of  $\mu$ . For the simulation  $\mu$  was chosen to be positive. The matched filter might be used if the mixture parameter were unknown. This would correspond to assuming that  $\varepsilon = 0$  when in fact  $\varepsilon > 0$ . A limiter of course would improve the performance. The results are shown in Figure 3. The Rao test allows one to detect signals 30 dB lower than a matched filter, clearly illustrating the importance of using the mixture parameter information in any detector.

## VII Conclusions

We have shown that for detection of known signals of unknown amplitude in non-Gaussian noise whose PDF is not completely specified that near optimal performance may be obtained. The optimality is an asymptotic optimality (large data records or small signals) but generally holds for problems of practical interest. The optimal detector is obtained by using a GLRT or a Rao test, the latter being easier to implement, and is based on the assumption that the PDF of the noise is symmetric. This condition is not overly restrictive in that nearly all real world noise has this property. Another benefit of the theory is that the detection performance is easily computed analytically. Future work will extend the results to the case of non-independent noise.

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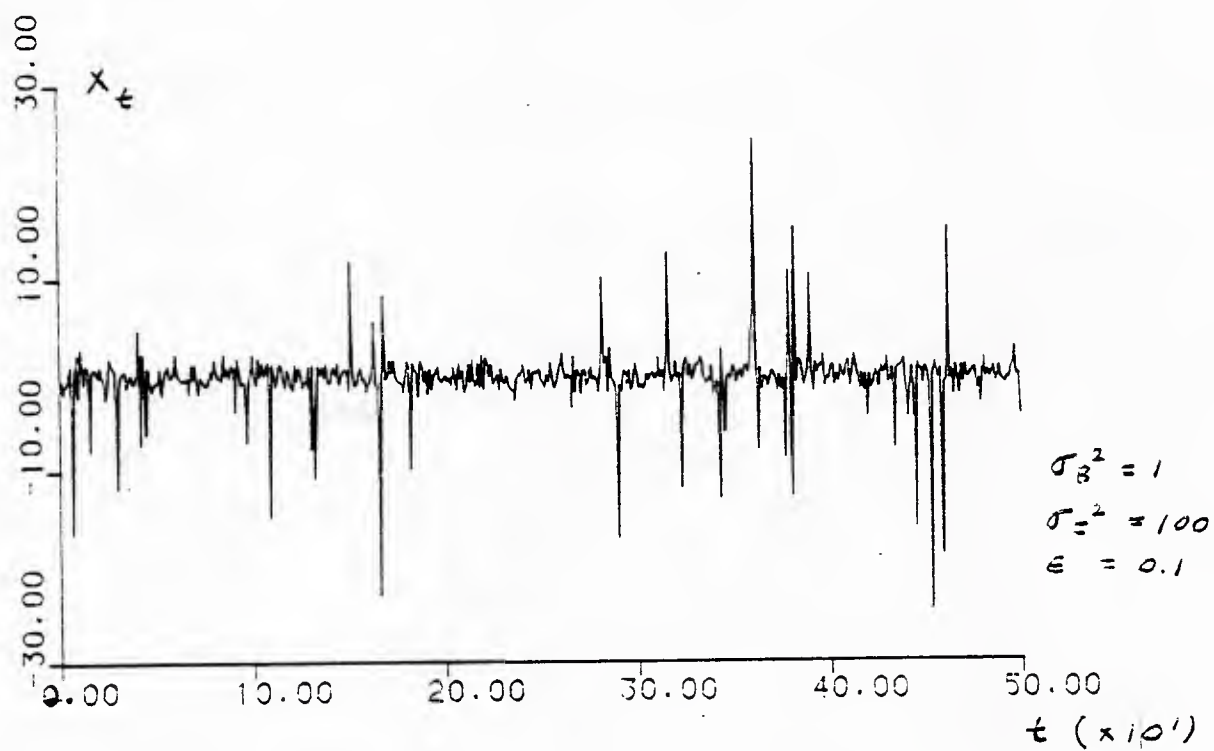


Figure 1 - Realization of Gaussian  
Mixture Noise

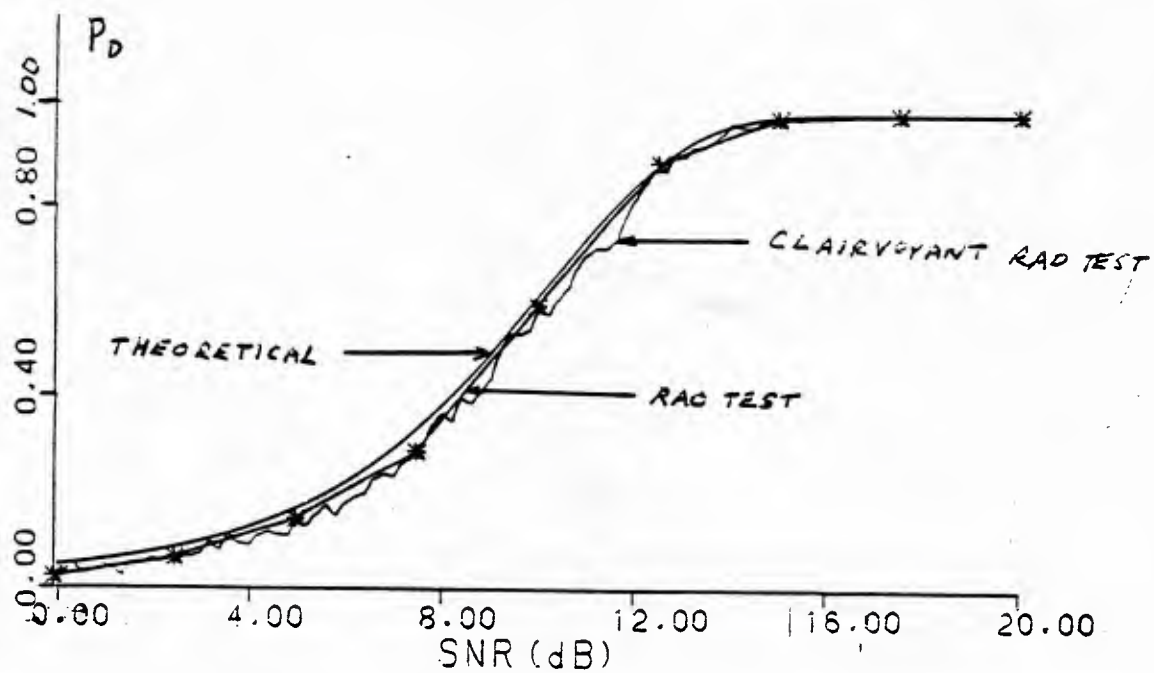


Figure 2 - Comparison of Detection Performance -  
Rao Test and Clairvoyant Rao Test

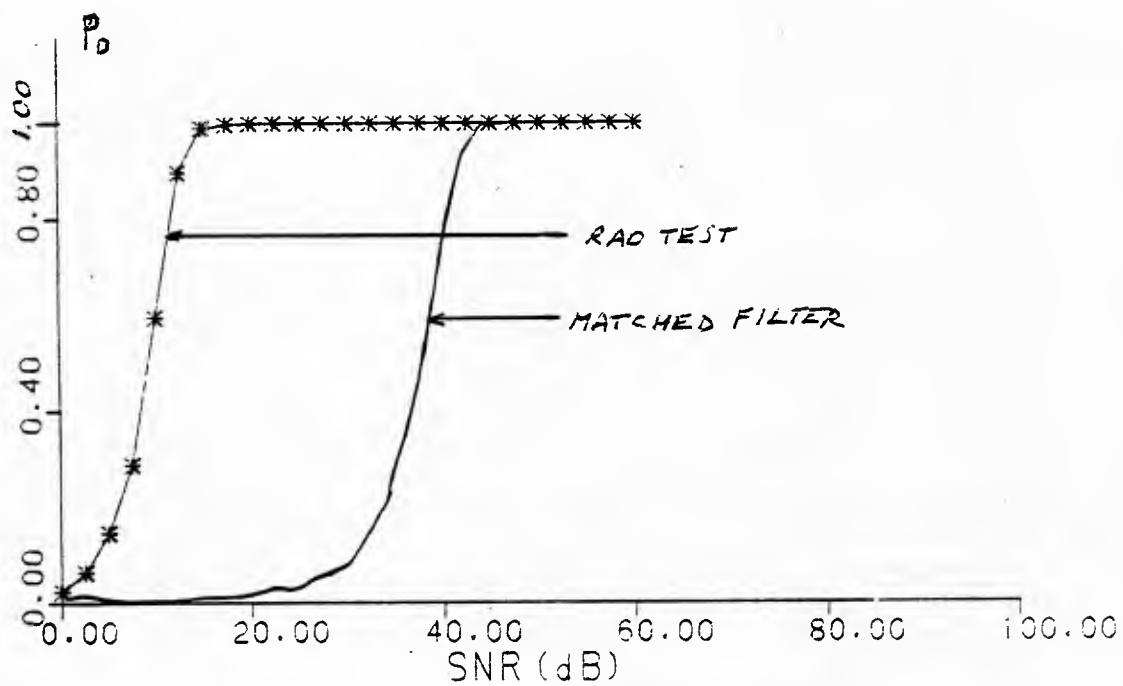


Figure 3 - Comparison of Detection Performance -  
Rao Test and Matched Filter

## Appendix A

### Example of GLRT Clairvoyant Performance

In this appendix the GLRT for two detection problems are derived. Consider the generic problem of

$$H_0 : x_t = n_t$$

$$H_1 : x_t = \mu + n_t$$

for  $t = 0, 1, \dots, N-1$ .

$\mu$  is an unknown amplitude either positive or negative.  $n_t$  is independent and identically distributed noise samples with each sample having the PDF  $n_t \sim N(0, \sigma^2)$ . In problem 1 the noise variance  $\sigma^2$  is assumed unknown while in problem 2,  $\sigma^2$  is known a priori. The GLRTs for problems 1 and 2 respectively are

$$L_G = \frac{\prod_{t=0}^{N-1} \frac{1}{\sqrt{2\pi \hat{\sigma}^2}} e^{-\frac{1}{2\hat{\sigma}^2} \sum_{t=0}^{N-1} (x_t - \hat{\mu})^2}}{\prod_{t=0}^{N-1} \frac{1}{\sqrt{2\pi \hat{\sigma}^2}} e^{-\frac{1}{2\hat{\sigma}^2} \sum_{t=0}^{N-1} x_t^2}} \quad (A-1)$$

and

$$L_{GC} = \frac{\prod_{t=0}^{N-1} \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2\sigma^2} \sum_{t=0}^{N-1} (x_t - \hat{\mu})^2}}{\prod_{t=0}^{N-1} \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2\sigma^2} \sum_{t=0}^{N-1} x_t^2}} \quad (A-2)$$

It is well known that for problem 1

$$\hat{\mu} = \frac{1}{N} \sum_{t=0}^{N-1} x_t = \bar{x}$$

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{t=0}^{N-1} (x_t - \bar{x})^2$$

$$\sigma^2 = \frac{1}{N} \sum_{t=0}^{N-1} x_t^2$$

and that for problem 2

$$\hat{\mu} = \bar{x}$$

Substitution of the MLEs into (A-1) and (A-2) yields

$$L_G = (\hat{\sigma}^2 / \sigma^2)^{N/2}$$

$$L_{GC} = e^{-\frac{1}{2\sigma^2} \left[ \sum_{t=0}^{N-1} (x_t - \bar{x})^2 - \sum_{t=0}^{N-1} x_t^2 \right]}$$

$$\text{Since } \hat{\sigma}^2 = \sigma^2 - \bar{x}^2,$$

$$L_G = \left[ 1 - \frac{\bar{x}^2}{\sigma^2} \right]^{N/2}$$

so that finally

$$2 \ln L_G = N \ln \left[ 1 - \frac{\bar{x}^2}{\frac{1}{N-1} \sum_{t=0}^{N-1} x_t^2} \right]$$

Also, after some simplification

$$L_{GC} = e^{-\frac{1}{2\sigma^2} N \bar{x}^2}$$

which yields

$$2 \ln L_{GC} = \frac{N \bar{x}^2}{\sigma^2} = \left[ \frac{\bar{x}}{\sigma/\sqrt{N}} \right]^2$$

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